Electric dipole

In these notes we want to look to the electric field generated by a dipole for very long distances. We remind that a dipole is composed by two equal and opposite charges $\pm q$. Our derivation is based on the computation of the associated electric potential, and then by taking derivatives. This shall also convince you of the advantages of working directly with the potential function.

1 Electric potential of the dipole

We consider a dipole where the $\pm q$ charges are located respectively at positions $\pm d/2$ on the z axis. Notice how the dipole is rotationally symmetric around the z axis; therefore we only need to find the potential in an arbitrary plane containing the line connecting the two charges.

Consider a point P with coordinates (r, θ) as shown in Figure 1. Let's call by r_+ the distance between the charge q and point P, and similarly r_- the distance between the charge $-q$ and P. We can write the exact expression of the potential at point P as

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{1}$$

We want now to derive an approximate form of the potential of the dipole at large distances, i.e. when $r \gg d$. We can use the law of cosines to express r_+ in terms of r and θ; we get

$$r_+ = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta}\,. \tag{2}$$

Since we want to perform explore the limiting behaviour $r \gg d$, the previous expression can be written in a more convenient form

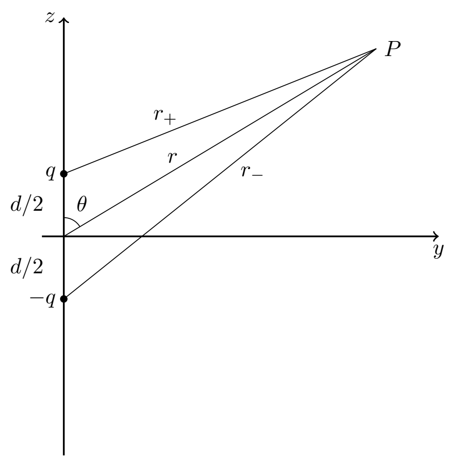

Figure 1: Electric dipole

$$r_+ = r\sqrt{1 - \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2}\,.$$

Now is evident that the second and third terms inside the square-root are small, since they involve the ratio $d/r \ll 1$. We can therefore Taylor expand the square-root at first order in d/r by using $(1+x)^a \simeq 1 + ax$ for $x \simeq 0$. We obtain

$$r_+ \simeq r - \frac{d}{2}\cos\theta\,. \tag{3}$$

Notice that we have neglected the term of order $(d/r)^2$ since it is subleading. The same steps can be repeated on r_-, the only difference being that the angle between the two known sides of the triangle (i.e. those of length r and $d/2$) is $\pi - \theta$:

$$r_- \simeq r - \frac{d}{2}\cos(\pi - \theta) = r + \frac{d}{2}\cos\theta\,. \tag{4}$$

We can now plug the expressions of r_+ and r_- inside equation (1) and Taylor expand again

$$\begin{aligned} V(r,\theta) &\simeq \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right) \simeq \frac{q}{4\pi\epsilon_0 r}\left[\left(1 + \frac{d}{2r}\cos\theta\right) - \left(1 - \frac{d}{2r}\cos\theta\right)\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2} \end{aligned} \tag{5}$$

Notice that potential decays as $1/r^2$ for large distances, differently to what happens to point charges. This type of behaviour will induce a decay of $1/r^3$ for the electric field, as we will see below. This makes sense: when looked from very far away two opposite charges look like a single neutral particle, so the the potential and the corresponding electric field should decay *faster* then the one corresponding to a single charge.

We can now define a vector called *dipole moment* as

$$\boldsymbol{p} = q\boldsymbol{d}\,, \tag{6}$$

where $\boldsymbol{d}$ is oriented by definition from the negative charge to the positive one. In terms of the dipole moment the potential of the dipole for long distances can be written as

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}\,. \tag{7}$$

2 Electric field from the potential

Since we know the potential, we can compute the electric field by using

$$\boldsymbol{E} = -\boldsymbol{\nabla} V\,. \tag{8}$$

At the present stage the potential is expressed in spherical coordinates r, θ and φ (the potential is independent of φ because as we anticipated, the dipole is rotationally symmetric around the z axis). Therefore we should know what are the components of the gradient in spherical coordinates (see Chapter 1 Griffiths):

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\boldsymbol{\varphi}} \tag{9}$$

Notice that V does not depend on φ, but only on θ and r, so the electric field lies in the plane $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ as expected. We get

$$E_r = -\frac{\partial V}{\partial r} = \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{r^3} \tag{10a}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\epsilon_0}\frac{p\sin\theta}{r^3} \tag{10b}$$

$$E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} = 0 \tag{10c}$$

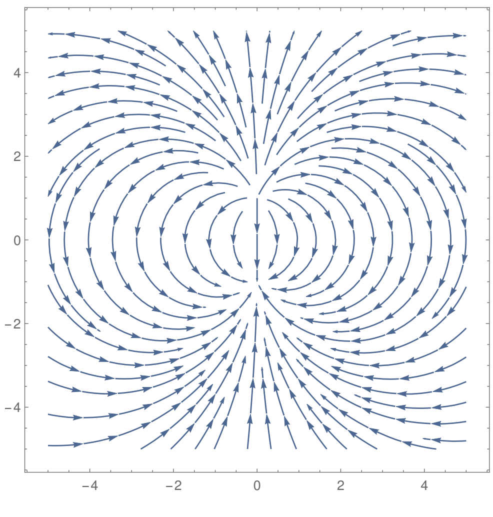

Figure 2: Field lines of a dipole with $d = 2$.

2.1 Electric field in cartesian components

We can also express (7) in cartesian coordinates and do some algebra. The potential of a dipole in cartesian coordinates reads

$$V(x, y, z) = \frac{p}{4\pi\epsilon_0} \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \tag{11}$$

since the dipole moment $\boldsymbol{p}$ points as $\hat{z}$. Taking partial derivatives we obtain:

$$E_x = -\frac{\partial V}{\partial x} = \frac{p}{4\pi\epsilon_0} \frac{3xz}{(x^2 + y^2 + z^2)^{5/2}} \tag{12a}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{p}{4\pi\epsilon_0} \frac{3yz}{(x^2 + y^2 + z^2)^{5/2}} \tag{12b}$$

$$E_z = -\frac{\partial V}{\partial z} = \frac{p}{4\pi\epsilon_0} \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} . \tag{12c}$$

Since $pz = \boldsymbol{p} \cdot \boldsymbol{r}$, we can rewrite the previous expressions in a much more compact form:

$$E_x = \frac{3}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^5} x \tag{13a}$$

$$E_y = \frac{3}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^5} y \tag{13b}$$

$$E_z = \frac{p}{4\pi\epsilon_0} \left(\frac{3z^2}{r^5} - \frac{1}{r^3} \right) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\boldsymbol{p} \cdot \boldsymbol{r}}{r^5} z - \frac{p}{r^3} \right) , \tag{13c}$$

or

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0} \left[\frac{3\boldsymbol{p} \cdot \boldsymbol{r}}{r^5} \boldsymbol{r} - \frac{\boldsymbol{p}}{r^3} \right] \tag{14}$$

The first term $\frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^5} \boldsymbol{r}$ gives a contribution to the electric field in the radial direction $\boldsymbol{r}$; the second term $\frac{\boldsymbol{p}}{r^3}$ instead is directed as the dipole moment. The electric field therefore lies in the plane zr. Notice also that both the terms in equation (14) decay as $1/r^3$ for large distances.

Let's look to some particular case:

- On the z axis (i.e. for $x = y = 0$ or in spherical coordinates $\theta = 0$, $r = z$), both $E_x = 0$ and $E_y = 0$. For the z component instead $E_z = \frac{p}{2\pi\epsilon_0} \frac{1}{z^3}$; the electric field at large distances is therefore directed as the dipole moment vector (when z is positive).

- In the xy plane (i.e. $z = 0$, $\boldsymbol{p} \cdot \boldsymbol{r} = 0$ or $\theta = \pi/2$ in spherical coordinates) we have again $E_x = 0$ and $E_y = 0$. In addition $E_z = -\frac{p}{4\pi\epsilon_0} \frac{1}{r^3}$ a result that we have already derived previously.

I stress that those expressions are valid only for $r \gg d$, so you are not allowed to use them for small distances if you do not want to run into gross errors. For example the true electric field on the z axis is always directed as the dipole moment apart for the region between the two charges and containing the origin, where the electric field is directed in the opposite to $\boldsymbol{p}$. This is obviously different to what predicts the large distance expansion.

For a plot of the field lines of the dipole, take a look at Figure 2.