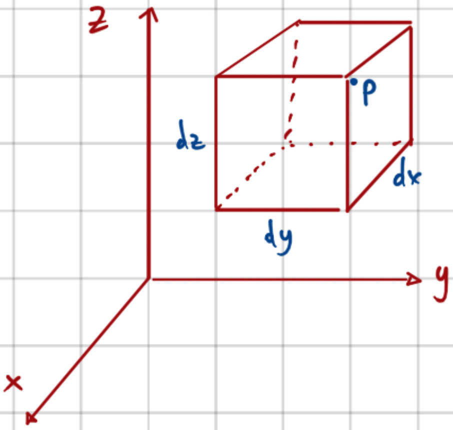
① Divergence Theorem

$$\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{S}$$

where $\vec{v}$ is a vector field, V is a volume and S its corresponding surface.

Sketch of the proof:



$$\vec{v}(x,y,z) = v_x(x,y,z)\,\hat{x} + v_y(x,y,z)\,\hat{y} + v_z(x,y,z)\,\hat{z}$$

Let's compute the (infinitesimal) fluxes of $\vec{v}$ through each of the 6 faces of a cube with (infinitesimal) sides dx, dy, dz

Start from the "back" face. Call the point in the middle of the "back" face as P

$$d\Phi_{back}(\vec{v}) \equiv \vec{v}(P) \cdot d\vec{S} = \vec{v}(P) \cdot \hat{n}\, d\vec{S} = \vec{v}(P) \cdot (-\hat{x}\, dS) = -v_x(P)\, dS = -v_x(P)\, dy\, dz$$

Let's do the same for "front" face. Call the point in the middle of the "front" face as P'

$$d\Phi_{front}(\vec{v}) = \vec{v}(P') \cdot d\vec{S} = \vec{v}(P') \cdot (\hat{x}\, dS) = v_x(P')\, dS = v_x(P')\, dy\, dz$$

However since P' is infinitesimally close to P, we can Taylor expand:

$$v_x(P') = v_x(P) + \frac{\partial v_x}{\partial x} dx + O(dx^2)$$

and the flux is

$$d\Phi_{front} = \left(v_x(P) + \frac{\partial v_x}{\partial x} dx \right) dy\, dz$$

The sum of the fluxes from front and back faces is

$$d\Phi_{front+back} = d\Phi_{front} + d\Phi_{back} = \left(v_x + \frac{\partial v_x}{\partial x} dx - v_x \right) dy\, dz$$

$$= \frac{\partial v_x}{\partial x} dx\, dy\, dz$$

We can repeat the same argument for the other 4 faces:

$$d\Phi_{top+bottom} = d\Phi_{top} + d\Phi_{bottom} = \frac{\partial v_y}{\partial y} dx\, dy\, dz$$

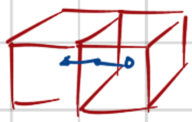
$$d\Phi_{left+right} = d\Phi_{left} + d\Phi_{right} = \frac{\partial v_z}{\partial z} dx\, dy\, dz$$

Therefore the flux out of an infinitesimal cube is

$$d\Phi_{cube} = d\Phi_{front+back} + d\Phi_{top+bottom} + d\Phi_{left+right}$$

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) dx\, dy\, dz = (\vec{\nabla} \cdot \vec{v}) \underbrace{dx\, dy\, dz}_{d\tau}$$

We can extend the reasoning above to a volume V by dividing it into small cubes. The total flux out of its boundary surface S is determined by summing the fluxes out of infinitesimally small cubes!



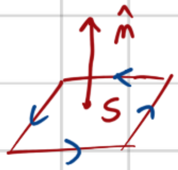
Notice that if we have two adjacent small cubes the sum of the fluxes out of the common side adds up to 0 (because the normal for one cube is opposite to the other!!)

② STOKES' THEOREM

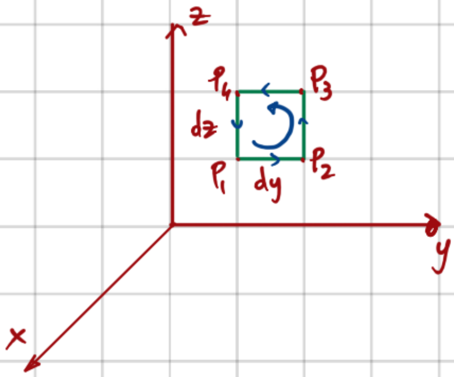
"the flux of the curl of $\vec{v}$ across a surface S is equal to the circulation of $\vec{v}$ along the boundary line C of S"

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{l}$$

The way of travelling C is given by the right-hand rule



Sketch of the proof : Let's take a square of sides of length dy and dz and vertices P_1, P_2, P_3, P_4. Let's compute the line integral of $\vec{v}$ along each side.



$$\overline{P_1P_2} : \quad \vec{v} \cdot d\vec{l} = \vec{v}(P_1) \cdot (\hat{y}\, dy) = v_y(P_1)\, dy$$

$$\overline{P_3P_4} : \quad \vec{v} \cdot d\vec{l} = \vec{v}(P_4) \cdot (-\hat{y}\, dy) = -v_y(P_4)\, dy$$

but $v_y(P_4) \simeq v_y(P_1) + \dfrac{\partial v_y}{\partial z} dz$

$$\overline{P_4P_1} : \quad \vec{v} \cdot d\vec{l} = \vec{v}(P_1) \cdot (-\hat{z}\, dz) = -v_z(P_1)\, dz$$

$$\overline{P_2P_3} : \quad \vec{v} \cdot d\vec{l} = \vec{v}(P_2) \cdot (\hat{z}\, dz) = v_z(P_2)\, dz$$

but $v_z(P_2) \simeq v_z(P_2) + \dfrac{\partial v_z}{\partial y} dy$

Now we add up all the contributions:

$$\overline{P_1P_2} + \overline{P_2P_3} + \overline{P_3P_4} + \overline{P_4P_1} = \left(v_y - v_y - \frac{\partial v_y}{\partial z}dz\right)dy + \left(-v_z + v_z + \frac{\partial v_z}{\partial x}dy\right)dz$$

$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)dy\,dz = (\vec{\nabla}\times\vec{v})_x\,dy\,dz$$

The same idea can be applied to an infinitesimal square of area $dx\,dz$ and of area $dy\,dx$