Electric current and the theory of circuits

We want here to discuss the motion of electric charges in conductors due to the effect of electric fields.

1 Electric current

Consider a double plate capacitor charged up with a potential difference ΔV. Between the plates of the capacitor there is an electric field $\boldsymbol{E}$ such that

$$\Delta V = V_+ - V_- = -\int_-^+ \boldsymbol{E} \cdot d\boldsymbol{l} \,. \tag{1}$$

where "+" and "−" are respectively the plates where positive and negative charge is located. Suppose now that we connect the two plates with a conducting wire.

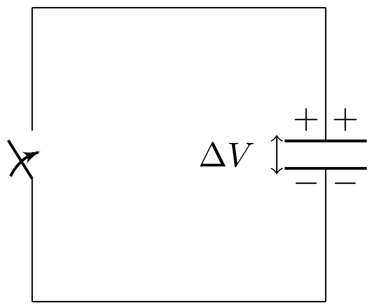


Figure 1: Capacitor connected to a conducting wire

When the two plates are connected, we observe the following:

- the potential difference ΔV decreases in time and goes to 0 exponentially fast; the same is observed for the electric field;
- the charge on the two plates tends to zero;
- the conducting wire heats up;
- the needle of a compass close to the wire starts to move.

All these phenomena can be justified by the fact that charges move from one plate to the other; we will equivalently say that some "electric current" has flowed from one plate to the other. In the following we want to understand how the first three observations can be physically justified (the last one will be only explained in the second part of the course, when we will deal with magnetic phenomena).

We define the *electric current* passing inside a conducting wire as the ratio between the charge dq that flows in the interval of time dt through a section S of the conductor and the interval of time dt itself:

$$I = \frac{dq}{dt} \,. \tag{2}$$

The base unit of electric current is the Ampère $1A = 1C/1s$. For historical reasons if some positive charges move into a certain direction, then we say that the current goes in the same direction as well. In other words, negative charges move in a direction opposite to the current. This can generate confusion since the particles that are responsible of the current (i.e. that really move through the wire) are the electrons.

A current is defined to be *steady* if it does not depend on time.

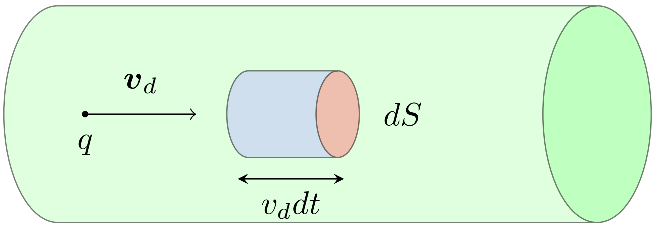


Figure 2: Here it is displayed a conductor wire where some current flows. The number of charges that pass through the surface element dS of the conductor (red circle) during time dt is equal to the number of charges that are located in the cylinder of base dS and height $v_d dt$.

2 Current density

As we had already seen a metallic conductor is composed by a regular three-dimensional structure of atoms with a large number of electrons (on average one electron per atom) that are *free* to move inside the conductor itself.

The number of electrons per unit volume that are free in a conductor depends on the material and on temperature. For copper at room temperature $T = 27°$, there are approximately $8 \cdot 10^{28}$ electrons per m^3.

In the absence of an electric field the electrons move with velocity $\boldsymbol{v}_T$ in *random directions*, only because of thermal energy. The modulus of the velocity of a single electron v_T can be computed by the equipartition theorem. This theorem can be proved by using statistical mechanics, and this goes beyond the scope of this course. However the expression of v_T is simple enough, and is valid not only for electrons but also for small particles of mass m:

$$v_T = \sqrt{\frac{3k_B T}{m}} \tag{3}$$

where $k_B \simeq 1.38 \cdot 10^{-23}$ J K^{-1} is Boltzmann's constant (K stands for *Kelvin*, the unit of temperature, connected to degrees by $1K = 1° + 273.15$). v_T increases when temperature increases or the mass of the particles decrease. At room temperature (i.e. $T \simeq 300$ K) the speed of the electrons is approximately $v_T \approx 120$ km/s (quite huge value!). The velocity of each electron however is directed in a random direction, so that on average no current is observed.

When an electric field is present however the electrons will move *collectively* in the direction *opposite* (because they possess negative charge) to the electric field itself with a so called *drift* velocity $\boldsymbol{v}_d$. This motion generates a current directed as the external electric field. We will derive an approximately good estimation of v_d in the next sections. Notice that even if the electrons are *accelerated* by the electric field, in practice they move at *constant* speed v_d along the direction of the electric field. Is this a contraction of Newton's second law? No, since we are neglecting the fact that electrons frequently collide with the atoms composing the conductor; it's like driving along a road but stopping at every traffic light! It is true that between two traffic lights we accelerate, but once a traffic light is met we abruptly stop and we need to start all over again from zero velocity. So your average speed is not increasing in time.

Consider now a conducting wire which has n free charges (each one of charge $q = -e$) per unit volume. Take also an infinitesimal element of surface $d\boldsymbol{S}$, whose normal is in the same direction as the drift velocity of the electrons. The charge that passes through the infinitesimal surface element dS during an infinitesimal time dt is

$$dq = nqv_d \, dS \, dt \tag{4}$$

since the charges that are able to pass through the surface element in a time dt are located at most at a distance $v_d dt$; therefore dq can be obtained by multiplying the charge per volume nq times the total volume of the cylinder of base dS and height $v_d dt$, see Figure 2.

In general, if the original surface element $d\boldsymbol{S}$ is inclined by an angle θ with respect to $\boldsymbol{v}_d$, all we have to do is to project it in the direction of the drift velocity: e.g. if $\boldsymbol{v}_d$ is orthogonal to $d\boldsymbol{S}$ no particle pass through it, see Figure 3.

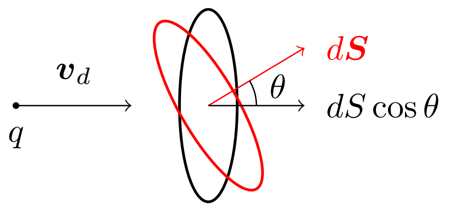


Figure 3

We can therefore generalize the previous equation to

$$dq = nq\boldsymbol{v}_d \cdot d\boldsymbol{S}dt \equiv \boldsymbol{J} \cdot d\boldsymbol{S}dt \tag{5}$$

where we have defined the current density as

$$\boldsymbol{J} = nq\boldsymbol{v}_d \tag{6}$$

whose dimension is $[A/m^2]$. Therefore the current dI that passes through an infinitesimal surface $d\boldsymbol{S}$ is

$$dI = \boldsymbol{J} \cdot d\boldsymbol{S}\,. \tag{7}$$

Integrating over a section of the conductor, we get that the current that flows through it is

$$I = \int_S \boldsymbol{J} \cdot d\boldsymbol{S} = \Phi_S(\boldsymbol{J})\,. \tag{8}$$

The previous equation tells us that the flux of the current density vector through the surface S gives us the current that flows through the section S itself.

3 Conservation of charge: continuity equation

We have already observed that the total electric charge must be conserved. We can express this idea mathematically by using the density of current vector $\boldsymbol{J}$.

Consider a fixed closed surface S where a certain amount of charge $q(t)$ at time t is contained. If in the time interval dt, the charge decreases by a quantity dq, since the total charge must be conserved, this means that the quantity of charge dq must be traversed through the surface S:

$$-\,dq = \oint_S \boldsymbol{J} \cdot d\boldsymbol{S}dt\,. \tag{9}$$

The sign on the left-hand side is needed since the flux is positive (charge is flowing out of the surface), but at the same time, charge inside the surface is decreased by a quantity $dq < 0$. Dividing by the time interval dt, we get

$$-\frac{dq}{dt} = \oint_S \boldsymbol{J} \cdot d\boldsymbol{S}\,. \tag{10}$$

Now let's call by V the volume enclosed by S and let be $d\tau = dxdydz$ an infinitesimal volume element. If we call by $\rho(x, y, z, t)$ the charge density at time t, we get

$$q(t) = \int_V \rho(x, y, z, t)\, d\tau \tag{11}$$

Since the surface S is fixed in time so is the volume V. We can thus differentiate the previous expression with respect to time and bring the time derivative inside the integral

$$\frac{dq}{dt} = \int_V \frac{\partial \rho}{\partial t}\, d\tau\,. \tag{12}$$

We can now use the divergence theorem on the right hand side of equation (10)

$$\oint_S \boldsymbol{J} \cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{J}\, d\tau \tag{13}$$

Combining (13) and (12) we get

$$-\int_V \frac{\partial \rho}{\partial t}\, d\tau = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{J}\, d\tau\,. \tag{14}$$

This relation is valid for every volume V we choose; therefore the equality of the integrals implies the equality of the integrands

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \tag{15}$$

This relation is known as the *continuity equation* and express in a differential form the conservation of charge.

In *stationary* conditions, no electric quantities depend on time. In particular this means that the density of charge is constant, so that

$$\frac{\partial \rho}{\partial t} = 0\,. \tag{16}$$

In those conditions we see that $\boldsymbol{J}$ is a solenoidal vector

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0\,. \tag{17}$$

4 Ohm's law

Consider a wire with cross-section S, length l. Let's apply a constant potential difference ΔV over the two sections of the wire. If we maintain the potential difference constant in time, as a result, a *steady electric current* will start flowing through the wire, from the section with higher potential (indicated with "+") to the lower one (indicated with "−" see Fig. 4). If the electric potential difference is

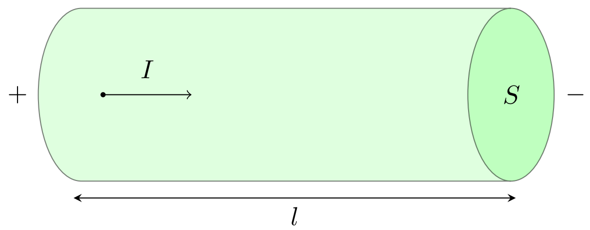


Figure 4

not so high as to cause electrical discharges that would damage the material irreversibly, it is found experimentally that there is a *linear* relationship between the potential difference and the current flowing between the two sections of the wire

$$\Delta V = RI\,. \tag{18}$$

The proportionality constant R is known as the *resistance* of the material. Equation (18) is known as *Ohm's law*; a material that satisfies it is also know as an *ideal resistor*[1]. The Ohm ($1\Omega = 1V/1A$) is the unit of electrical resistance. Its magnitude depends only on the geometry of the material (length, shape...) and on the physical conditions in which the resistor is placed (most importantly: temperature). The symbol of a resistor as usually used in circuit analysis is depicted in Fig. 5.

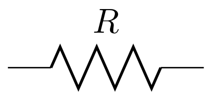


Figure 5: Symbol of a resistor R.

4.1 Microscopic justification of Ohm's law

We want here to derive Ohm's law in a very crude way, using some simple considerations based on classical mechanics.

To accurately describe the phenomenon of electric current flow in solids, one should use quantum mechanics instead. However by applying the laws of classical mechanics to the electrons, we can still obtain some insight and some qualitative, approximate description of the phenomenon. The simple model of conducting electrons we want to describe is called *Drude model*.

As we have anticipated before, the electrons in a metallic wire move with a velocity v_T in random directions, due to temperature. During their motion, the electrons collide with the atoms of the metal. Let us denote by τ the (average) time between two successive collisions. In copper at room temperature ($T = 300$ K) $\tau \approx 2.5 \cdot 10^{-14}$ s.

When we apply a constant potential difference ΔV to the two sections of the wire, we generate a constant electric field[2] given by $E = \Delta V/l$, and each electron will experience a force given in modulus by

$$F = eE = m_e a\,, \tag{19}$$

where $e \simeq 1.6 \cdot 10^{-19}$ C is the (modulus) of the charge of the electron. The corresponding acceleration experienced by the electron between two successive collision will be

$$a = \frac{F}{m_e} = \frac{eE}{m_e}\,, \tag{20}$$

where $m_e = 9.1 \cdot 10^{-31}$Kg is the mass of the electron. The speed that the electrons will acquire between the collisions is the *drift* velocity we have introduced previously and is given by

$$v_d = a\tau = \frac{e\tau}{m_e}E\,. \tag{21}$$

You see that v_d is proportional to the electric field and the time between collisions: if E or τ are larger the drift velocity will increase as well. To get an idea of the magnitude of the drift velocity, consider that if I apply a potential difference $\Delta V = 1$ V to a piece of copper of length $l = 1$ m, the corresponding electric field will be $E = \Delta V/l = 1$ V/m, and the drift velocity would be approximately equal to[3] $v_d \approx 4.8 \cdot 10^{-3}$ m/s.

We know from equation (4) that the current that flows through a cross section S of the wire is given by

$$I = nev_d S\,, \tag{22}$$

[1]In this course we will always deal with ideal resistors. For the sake of simplicity we will always denote them as "resistors". Resistors are usually made of *poorly* conducting materials. For a perfect conductor the resistance R is so low that it suffices a very small potential difference (and therefore a very small electric field) to have some current flowing inside it. See also the discussion in Griffiths, Section 7.1.1 (before example 7.1)

[2]Actually proving that the electric field is constant and uniform within the wire is not easy. See example 7.3 for an argument based on Laplace's equation.

[3]This result is surprising at first sight since v_d is very small, also compared to the velocity that the electron have due to thermal motion ($v_T \gg v_d$). How can we observe right after we apply a potential difference that some current flows in the wire if the electron have a so small drift velocity? If we could track an electron near one end of the wire, it would take approximately 200 seconds to reach the other end! The point is that the electric field propagates instantaneously (relative to the length scale of the wire) through the wire, so that *each* of the electrons in the metal will start moving with a drift velocity v_d!

where we remind that n is the charge per unit volume. Substituting the expression of the drift velocity we obtain

$$I = \sigma S E \tag{23}$$

where σ is called *conductivity* of the material and is given by

$$\sigma \equiv \frac{ne^2\tau}{m_e}\,. \tag{24}$$

Since $E = \frac{\Delta V}{l}$ we have

$$\Delta V = \left(\frac{l}{\sigma S}\right) I \tag{25}$$

which is Ohm's law, with

$$R = \frac{l}{\sigma S}\,. \tag{26}$$

Sometimes it is equation (23) that is called Ohm's law; in vectorial form it reads (noticing that I/S is the current density J)

$$\boldsymbol{J} = \sigma \boldsymbol{E}\,. \tag{27}$$

5 Joule's heating law

The movement of charges inside a conductor takes place because an electric field $\boldsymbol{E}$ acts on them. The electric field clearly has to do some work to move those charges in the conductor. We know that the work that the electric field must do to move an infinitesimal charge dq from point A at potential V_A and point B with potential V_B is

$$dW = (V_A - V_B)dq \equiv \Delta V\,dq\,. \tag{28}$$

If we consider a conducting wire in which it flows some current I, by definition the charge dq that flows in time dt through a section of the wire is $dq = Idt$. Therefore

$$P = \frac{dW}{dt} = \Delta V I \tag{29}$$

where P is the work per unit time (unit of measure: Watts). This is called *Joule's* law. If the conducting wire satisfies Ohm's law (18), then we can also recast Joule's law in other two equivalent forms

$$P = RI^2 = \frac{\Delta V^2}{R}\,. \tag{30}$$

Joule's law is also know as "heating" law since the energy that those charges acquire because of the electric field is converted into *heat* in the resistor as the result of the many collisions between the electrons and the nuclei composing the conductor.

6 Electromotive force

Batteries or *generators of electromotive force* are electric components that are able to maintain a constant potential difference between the ends of a current-carrying conductor.

In simple circuits the steady currents I is sustained (out of the battery) by the electrostatic field $\boldsymbol{E}_s$ that is generated by the accumulations of charges on the terminals of the battery. Those accumulations of charges must be continuously restored by the generator itself to the extent necessary to maintain a constant flow of electrical charges in the circuit over time. Therefore, in order to have steady currents in a circuit, the battery needs to produce a force per unit charge $\boldsymbol{E}_b$ that is able to sustain the current. $\boldsymbol{E}_b$ is usually confined

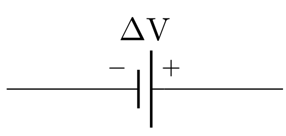


Figure 6: Symbol of a battery with potential difference ΔV between its positive and negative terminals.

in only one portion of the circuit, i.e. inside the battery. $\boldsymbol{E}_b$ has the role of separating the charges *inside the battery*, pushing the electrons towards the negative terminal of the battery fighting against the electrostatic field $\boldsymbol{E}_s$. The physical phenomena that generate $\boldsymbol{E}_b$ can be of very different nature: in a usual battery it is generated by chemical forces, in piezoelectric crystals it is the mechanical pressure that it is converted into an electrostatic field, etc...

The total field in the circuit is

$$\boldsymbol{E} = \boldsymbol{E}_s + \boldsymbol{E}_b\,. \tag{31}$$

We define the *electromotive force* or *emf* as the integral of the total electric field over the closed line identified by the circuit itself:

$$\mathcal{E} = \oint \boldsymbol{E}\cdot d\boldsymbol{l} = \oint \boldsymbol{E}_b\cdot d\boldsymbol{l}\,. \tag{32}$$

where we have used the fact that the electrostatic field $\boldsymbol{E}_s$ is conservative, so it does not contribute to a line integral over a closed loop.

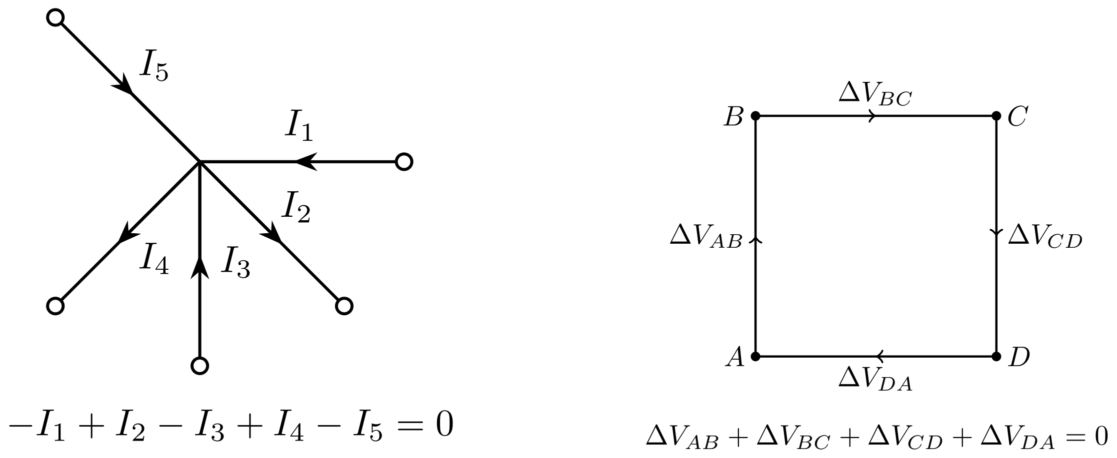
In an ideal battery[4] the net force on a charge is zero, so $\boldsymbol{E}_b = -\boldsymbol{E}_s$. It follows that the potential difference between the two terminals of the battery (call them A and B) is therefore equal to the emf

$$\Delta V = -\int_A^B \boldsymbol{E}_s\cdot d\boldsymbol{l} = \int_A^B \boldsymbol{E}_b\cdot d\boldsymbol{l} = \oint \boldsymbol{E}_b\cdot d\boldsymbol{l} = \mathcal{E}\,, \tag{33}$$

where we have used that $\boldsymbol{E}_s = 0$ outside the battery. Because $\mathcal{E}$ is the line integral of $\boldsymbol{E}_s$ between the terminals of the battery, the emf can be interpreted as the work done per unit charge.

7 Kirchhoff's laws

We are ready to state Kirkhoff's laws.



(a) First Kirchhoff's law

(b) Second Kirchhoff's law

Figure 7: Kirchhoff's laws

7.1 First Kirchhoff's law

In stationary conditions the sum of all the currents flowing into a node is equal to the sum of all the currents flowing out of that node.

In formulas

$$\sum_k I_k = 0\,, \tag{34}$$

[4] i.e. a battery that does not have an internal resistance

where we are considering the currents I_k to be positive if the charge is flowing out of the node, and negative otherwise if it is flowing towards the node, see the example in Fig. 7a.

This result follows from the fact that, in stationary conditions the current density vector $\boldsymbol{J}$ is a solenoidal field (17). So first Kirchhoff's law is a consequence of the conservation of charge.

7.2 Second Kirchhoff's law

The directed sum of the potential differences (voltages) around any closed loop is zero.

In formulas

$$\sum_k \Delta V_k = 0\,. \tag{35}$$

see as an example Fig. 7b. This law follows trivially from the the fact that the electrostatic field is conservative.

Remark: Kirchhoff's laws are valid even if the current is not constant in time, but it is constant across the whole circuit for each time t. This condition is known as "*quasi-static*" (or quasi-stationary) condition. A situation where the current depend on the position appears when the circuit is so big that the electric field (that travels at the speed of light) will take a non-negligible time to communicate its presence in distant parts of the circuit. In our cases the quasi-static condition can be considered to be always approximately true.

8 Resistors in series and in parallel

As in the case of capacitors we can combine two resistors in a circuit in two different ways: in series and in parallel.

8.1 Resistors in series

The key property of two resitors R_1, R_2 in series (see Fig. 8a) is that the same current flows through them. We can therefore write, referring to Fig. 8a, the following two equations for each resistance:

$$V_A - V_C = R_1 I \tag{36a}$$

$$V_C - V_B = R_2 I \tag{36b}$$

Summing these two equations we get

$$V_A - V_B = I(R_1 + R_2) = RI\,, \tag{37}$$

Therefore we conclude that two resistors in series behave like a single resistor of resistance

$$R = R_1 + R_2\,. \tag{38}$$

8.2 Resistors in parallel

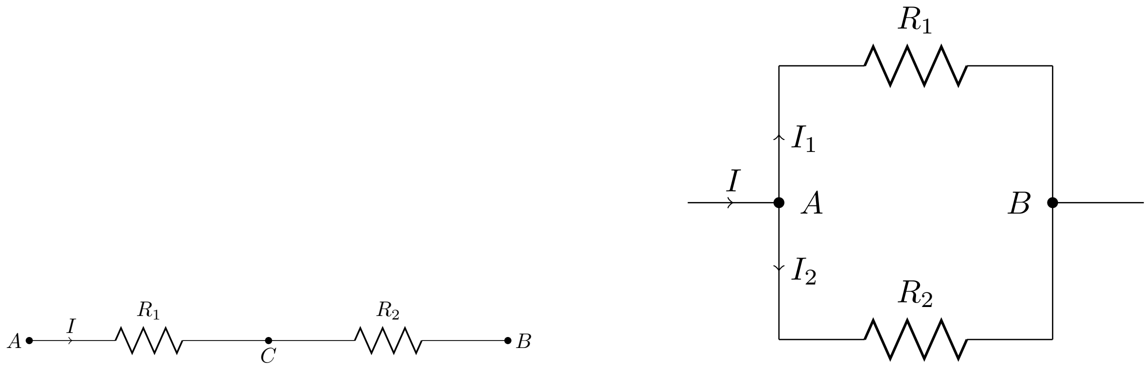
For two resistors in parallel (see Fig. 8b), we have that the potential difference at the endpoints of the two resistance is the same. Applying first Kirchhoff's law at node A we get that $I = I_1 + I_2$. We can write Ohm's law for each resistance as

$$I_1 = \frac{V_A - V_B}{R_1} \tag{39a}$$

$$I_2 = \frac{V_A - V_B}{R_2}\,. \tag{39b}$$

Summing the equations we get

$$I_1 + I_2 = I = (V_A - V_B)\left(\frac{1}{R_A} + \frac{1}{R_B}\right) = \frac{V_A - V_B}{R}\,, \tag{40}$$



(a) Series (b) Parallel

Figure 8: Resistors in series (left) and in parallel (right).

and so the equivalent resistance is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\,. \tag{41}$$

9 Discharge of a capacitor

Suppose that we have a circuit of the type represented in the Fig. 9, i.e. a charged capacitor C connected to a resistance R. For $t < 0$ the circuit is open. At $t = 0$ we close the circuit; what will happen to the charge on the capacitor? What's the behaviour in time of the charge on the capacitor?

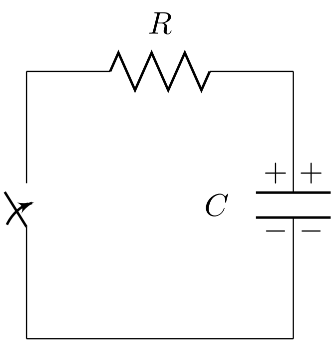


Figure 9: RC circuit with no battery.

Denote by $V(t)$ the potential difference at time t on the plates of the capacitor. For every time t we can write Ohm's law

$$V(t) = RI(t)\,. \tag{42}$$

Since the potential on the plates of a capacitor is related to the capacitance as $V(t) = Q(t)/C$ we have

$$\frac{Q(t)}{C} = RI(t)\,. \tag{43}$$

During the infinitesimal time interval dt a charge Idt flows through the resistance R. By the conservation of charge this should also be the decrease of the charge Q possessed by the capacitor, i.e. $-dQ = Idt$ so that $-\frac{dQ}{dt} = I$. Substituting the current inside (43) we get a differential equation of first order for the charge on the capacitor as a function of time:

$$\frac{dQ}{dt} = -\frac{Q}{RC}\,.$$

Denoting by $Q(t = 0) = Q_0$ the initial charge on the plates of the conductors that is know, the solution to the previous differential equation is

$$Q(t) = Q_0 e^{-t/\tau} \tag{44}$$

where $\tau = RC$. So the charge on the plates of the capacitor decrease exponentially rapidly in time with a time scale given by $\tau = RC$. If τ is large (i.e. if R or C or both are large), than the capacitor will take a larger time to completely discharge. We can also compute the behavior of the current

$$I(t) = \frac{Q_0}{\tau} e^{-t/\tau} \tag{45}$$

and from (42) the potential as function of time. As we have anticipated in section 1, both decrease exponentially in time.

What is the energy dissipated by the resistance during the discharge of the capacitor? During time dt the energy dissipated is $dW = RI(t)dt$, so that we have

$$W = R \int_0^{+\infty} I^2(t)dt = \frac{RQ_0}{\tau^2} \int_0^{+\infty} e^{-2t/\tau} dt = \frac{Q_0^2}{2C} \, . \tag{46}$$

Notice that this is the same energy that was stored in the capacitor before the circuit was closed.