Electric fields of discrete and continuous distribution of charge

In these notes we will look to simple examples of computations of electric fields.

1 Discrete Distributions

We will start by the discrete distribution case. Recall that the force on a "test" charge Q generated by a "source" charge q is given by Coulomb's law:

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathfrak{r}^2}\hat{\boldsymbol{\mathfrak{r}}} \tag{1}$$

where $\boldsymbol{\mathfrak{r}}$ is the separation vector from $\boldsymbol{r}'$ (the location of q) to $\boldsymbol{r}$ (the location of Q)

$$\boldsymbol{\mathfrak{r}} = \boldsymbol{r} - \boldsymbol{r}' \tag{2}$$

and $\hat{\boldsymbol{\mathfrak{r}}}$ is the corresponding versor. The electric field generated by q on position $\boldsymbol{r}$ is defined to be the force per unit charge

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{\boldsymbol{F}}{Q} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathfrak{r}^2}\hat{\boldsymbol{\mathfrak{r}}}. \tag{3}$$

If we have more than one source charge, the electric field is computed simply summing the individual electric fields of the sigle source charges (*superposition principle*).

1.1 Electric dipole (Problem 2.2 Griffiths)

We want to find the electric field (magnitude and direction) at a distance z above the midpoint between equal and opposite charges ($\pm q$), a distance d apart. Such a configuration of charges is called *electric dipole*.

The electric fields generated by the single charges are displayed in red in Figure 1. The sum of the two is directed as $\hat{x}$, so the vertical component cancels. We therefore have

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{\mathfrak{r}^2}\sin\theta\,\hat{\boldsymbol{x}}$$

and since $\sin\theta = \frac{d}{2\mathfrak{r}}$ with $\mathfrak{r} = \sqrt{z^2 + \left(\frac{d}{2}\right)^2}$ we get

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(z^2 + \left(\frac{D}{2}\right)^2\right)^{3/2}}\hat{\boldsymbol{x}}\,. \tag{4}$$

Check: if we set $d \to 0$ we correctly have $\boldsymbol{E} = 0$ (field generated by zero charge). Notice also that, for long distances $z \gg d$ the field goes like $\boldsymbol{E} \simeq \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\hat{\boldsymbol{x}}$, i.e. it decays as z^{-3} (not as z^{-2} as a single point charge).

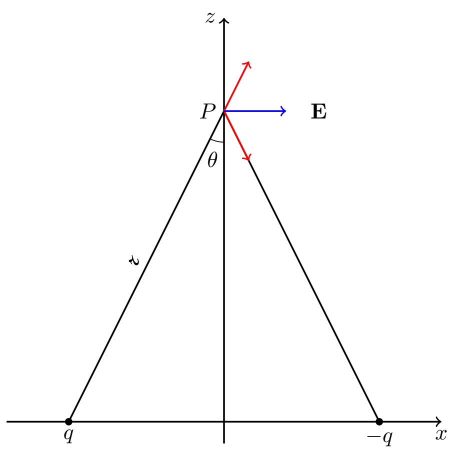

Figure 1: Electric dipole

2 Continuous charge distribution

Remind that if the charge is distributed over some continuous region (call it $\mathscr{R}$), the electric field is obtained by integrating over all the electric fields generated by infinitesimal small elements of charge dq

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathscr{R}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dq\,.$$

Depending on the region $\mathscr{R}$, the infinitesimal charge dq can then be expressed as a function of a density of charge. There are three cases:

- the charge is spread out along a line, with a charge per unit length given by $\lambda(\boldsymbol{r}')$. Then $dq = \lambda(\boldsymbol{r}')dl'$ where dl' is the infinitesimal line element and

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathscr{R}} \frac{\lambda(\boldsymbol{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}dl'\,. \tag{5}$$

- the charge is distributed along a surface, with a charge per unit area given by $\sigma(\boldsymbol{r}')$. Then $dq = \lambda(\boldsymbol{r}')dl'$ where da' is the infinitesimal element of area of the surface and

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathscr{R}} \frac{\sigma(\boldsymbol{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}da'\,. \tag{6}$$

- the charge fills a volume, with a charge per unit volume given by $\rho(\boldsymbol{r}')$. Then $dq = \rho(\boldsymbol{r}')d\tau'$ where $d\tau'$ is the infinitesimal element of volume and

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathscr{R}} \frac{\rho(\boldsymbol{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}d\tau'\,. \tag{7}$$

2.1 Electric field of a straight line

Consider a uniform distribution of charge λ on a straight line (i.e. infinitely long). We can compute the electric field at a point P which is located at a distance z from the closest point O on the line.

Take a line element dl_1 distant $\mathscr{r}$ from point P; for every element you take, there always exist another one (call it dl_2) which is symmetrical with respect to point O (i.e. it has the same distance $\mathscr{r}$ from point P), see Figure 2.

Those two line elements generate two infinitesimal electric fields $d\boldsymbol{E}_1$, $d\boldsymbol{E}_2$. Since dl_1 and dl_2 are symmetrical, the electric field $d\boldsymbol{E} = d\boldsymbol{E}_1 + d\boldsymbol{E}_2$ is orthogonal to the line, i.e. it is directed as $\hat{\boldsymbol{z}}$:

$$d\boldsymbol{E} = (dE_1 + dE_2)\cos\theta\,\hat{\boldsymbol{z}} = \frac{2\cos\theta}{4\pi\epsilon_0}\frac{\lambda dl}{\mathscr{r}^2}\hat{\boldsymbol{z}} \tag{8}$$

where we used Coulomb's law $dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda dl}{\mathscr{r}^2}$, and the fact that $dE_1 = dE_2$.

Now we have to integrate over all line elements. We therefore express the generic element dl and r as functions of the angle θ and of the distance z (which is constant):

$$\mathscr{r} = \frac{z}{\cos\theta} \tag{9a}$$

$$l = z\tan\theta\,. \tag{9b}$$

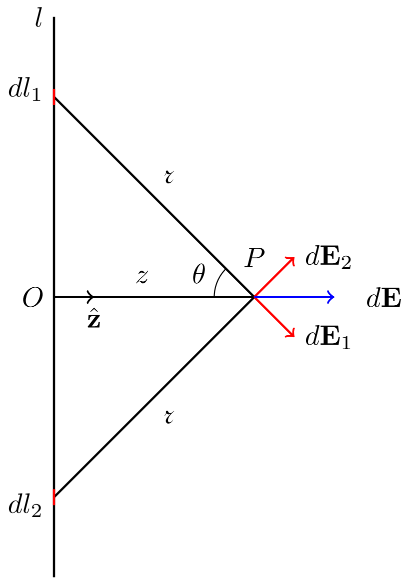

Figure 2: Line of charge

Since z is constant, we can differentiate the second equation: $dl = z\frac{d\theta}{\cos^2\theta}$. Substituting inside equation (8) and integrating over θ (from 0 to $\pi/2$) we obtain

$$\boldsymbol{E} = \int d\boldsymbol{E} = \frac{\hat{\boldsymbol{z}}}{2\pi\epsilon_0}\lambda\int_0^{\pi/2}\frac{\cos^3\theta}{z^2}\frac{z}{\cos^2\theta}d\theta = \frac{\hat{\boldsymbol{z}}}{2\pi\epsilon_0}\frac{\lambda}{z} \tag{10}$$

2.2 Uniformly charged ring (Problem 2.5 Griffiths)

Consider now a uniformly charged ring. We call by r the radius of the ring, $\hat{\boldsymbol{z}}$ the versor pointing as the axis of the ring. We want to compute the value of the electric field on a point P on the axis of the ring, and distant z from the center of the ring.

Take a element of length dl; the infinitesimal electric field that it generates is given by Coulomb's law

$$d\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dl}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,. \tag{11}$$

Of course the direction of the electric field depends on the element dl you choose. However for every dl you choose there is a corresponding element on the opposite side of the ring, which cancel every contribution of the electric field orthogonal to the $\hat{z}$ axis. The total electric field is therefore directed as $\hat{\boldsymbol{z}}$; we can compute it by projecting $d\boldsymbol{E}$ on the $\hat{z}$ axis and then summing over all the contributions coming from the ring

$$\boldsymbol{E} = \int d\boldsymbol{E}_z = \int \cos\theta d\boldsymbol{E} = \frac{\hat{\boldsymbol{z}}}{4\pi\epsilon_0}\int \frac{\lambda\cos\theta}{\mathscr{r}^2}\,dl \tag{12}$$

Notice now that, for every element dl you choose, both θ and $\mathscr{r}$ are the same. Since also λ is constant, the integral is trivial $\int dl = 2\pi R$ and we have

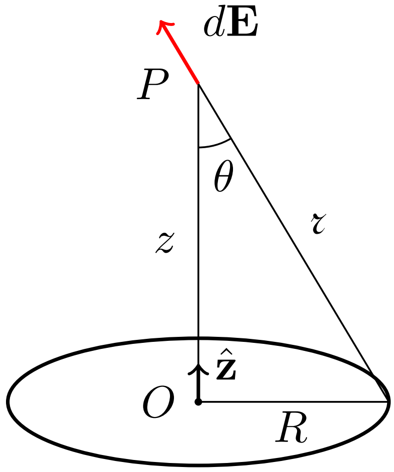

Figure 3: Charged Ring

$$\boldsymbol{E} = \frac{\hat{\boldsymbol{z}}}{4\pi\epsilon_0}\frac{\lambda(2\pi R)\cos\theta}{\mathscr{r}^2} = \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi R)z}{(z^2+R^2)^{3/2}}\hat{\boldsymbol{z}} \tag{13}$$

where we have used $\cos\theta = \frac{z}{\mathscr{r}}$ and $\mathscr{r}^2 = R^2 + z^2$. Notice that the electric field vanishes at the center of the ring (i.e. for $z = 0$). On the other hand, for long distances $z \gg R$ we obtain

$$E = \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi R)}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\,, \tag{14}$$

i.e. the ring has the same electric field of a point charge (indeed, $q = \lambda(2\pi R)$ is the total charge of the ring).

2.3 Uniformly charged disk (Problem 2.6 Griffiths)

Find the electric field a distance z above the center of a flat circular disk of radius R that carries a uniform surface charge σ. What does the formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

Since we know what is the field generated by a uniformly charged ring a distance z above the center, we can think to divide the disk into rings of radius r and thickness dr. The total charge of a ring is given by $dq = \sigma(2\pi r)dr$. The corresponding electric field is given by E

$$d\boldsymbol{E}_{\text{ring}} = \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi r)z}{(z^2+r^2)^{3/2}}\hat{\boldsymbol{z}}\,dr \tag{15}$$

The electric field of the disk will be obtained by summing all the electric fields of the rings having radius $r \in [0, R]$

$$\boldsymbol{E}_{\text{disk}} = \int d\boldsymbol{E}_{\text{ring}} = \frac{\hat{\boldsymbol{z}}}{4\pi\epsilon_0}\int_0^R \frac{\sigma(2\pi r)z}{(z^2+r^2)^{3/2}}\,dr = \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{|z|} - \frac{1}{\sqrt{z^2+R^2}}\right]\hat{\boldsymbol{z}}\,. \tag{16}$$

Notice how the quantity in square bracket is positive definite, so that the field is directed as $\hat{\boldsymbol{z}}$ above the circular disk (if we consider instead the field below the plane, i.e. for $z < 0$ the field is directed in the opposite was as $\hat{\boldsymbol{z}}$). When $R \to \infty$, the disk becomes a uniformly charged plane; since the second term goes to 0, we get

$$\boldsymbol{E}_{\text{plane}} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}}\,. \tag{17}$$

where $\hat{\boldsymbol{n}}$ is a unit vector that is directed out of the plane. Notice that the field generated by a uniformly charged plane is constant in whole space!

For $z \gg R$ we can Taylor expand

$$\frac{1}{\sqrt{z^2+R^2}} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} \simeq \frac{1}{z}\left(1-\frac{R^2}{2z^2}\right) \tag{18}$$

so that

$$\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}} \simeq \frac{R^2}{2z^3}$$

and

$$\boldsymbol{E} \simeq \frac{\sigma}{2\epsilon_0}\frac{R^2}{2z^2} = \frac{1}{4\pi\epsilon_0}\frac{\sigma\pi R^2}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2} \tag{19}$$

where $q = \sigma\pi R^2$ is the total charge of the disk. Again, for large distances from the center, the disk behaves as a point of charge q.