Additional notes on Poisson's and Laplace's equations

The task of electrostatics consists in finding the electric field of a fixed distribution of charge ρ. We have seen that the electrostatic field satisfies the two equations

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0 \tag{1b}$$

the first being Gauss's law in local form and the second one expressing the fact that the electrostatic field is conservative. The second equation implies the existence of a scalar function V (the potential) which is related to $\mathbf{E}$ by

$$\mathbf{E} = -\boldsymbol{\nabla} V \tag{2}$$

We have already seen that being a scalar quantity, V is in general easier to compute than the electric field. Combining the first equation of (1) with (2), we get

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}\,. \tag{3}$$

which is called *Poisson's equation*. If we are interested in finding the potential in a region where there is no density of charge, then Poisson's equation reduces to *Laplace's equation*

$$\nabla^2 V = 0\,. \tag{4}$$

The whole electrostatic problem can be recast in finding a way to solve Poisson's or Laplace's equations. This is in general an arduous task. However it is important to understand some properties of their solutions. In the following we will firstly see under which condition the solution to Poisson's equation is unique, and then we will prove one important property of the solutions of Laplace's equations.

1 Uniqueness theorem for Poisson's equation

Consider a region of space τ delimited by a surface S. Let $\rho(x, y, z)$ be the charge density through the region τ. If the potential assumes a value $V = V_S$ over the surface S (the so called "boundary condition") then the solution of the Poisson's equation (3) is unique.

Proof: Suppose there were two functions V_1 and V_2 satisfying Poisson's equation with the corresponding boundary condition, i.e.

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} \qquad \text{with } V_1 = V_S \text{ over } S \tag{5a}$$

$$\nabla^2 V_2 = -\frac{\rho}{\epsilon_0} \qquad \text{with } V_2 = V_S \text{ over } S\,. \tag{5b}$$

Now take the difference of the two: $V \equiv V_1 - V_2$. Therefore

$$\nabla^2 V = \nabla^2 (V_1 - V_2) = 0 \qquad \text{with } V = 0 \text{ over } S \tag{6}$$

i.e. V satisfies Laplace's equation with boundary condition $V = 0$ over S. Now we need to prove that the only function V with those properties is identically zero over all τ.

Consider the quantity $\boldsymbol{\nabla} \cdot (V\, \boldsymbol{\nabla} V)$ and integrate it over the volume τ. Using the divergence theorem we get

$$\int_\tau \boldsymbol{\nabla} \cdot (V\, \boldsymbol{\nabla} V) d\tau = \int_S V\, \boldsymbol{\nabla} V \cdot d\mathbf{S} = 0 \tag{7}$$

since $V = 0$ over S. On the other hand

$$\boldsymbol{\nabla} \cdot (V\, \boldsymbol{\nabla} V) = V\, \nabla^2 V + (\boldsymbol{\nabla} V)^2 = (\boldsymbol{\nabla} V)^2 \tag{8}$$

where we have used that $\nabla^2 V = 0$. So the left hand side of equation (7) gives

$$\int_\tau \boldsymbol{\nabla} \cdot (V\, \boldsymbol{\nabla} V) d\tau = \int_\tau (\boldsymbol{\nabla} V)^2 d\tau = 0\,. \tag{9}$$

What the previous equation tells us is that the function $(\boldsymbol{\nabla} V)^2$ is a positive or null quantity and its integral over τ vanishes. The only way the integral can vanish is therefore that $\boldsymbol{\nabla} V = 0$ i.e. $V = \text{const}$. However since $V = 0$ over S it must be that $V = 0$ over the whole region of space τ.

1.1 Solving the rest

If a powerful mathematician is able to solve Poisson's equation and hands you a solution V, then you can compute all the physically interesting quantities. Of course from (2) you can compute the electric field in the whole space τ and so you have access to the force that a particle with charge Q is subjected to by $\mathbf{F} = Q\mathbf{E}$.

By knowing $\mathbf{E}$ we can also compute the charge residing on a conductor. Remind that the field immediately outside a conductor is given by

$$\mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \tag{10}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface of the conductor at that location in space. By using equation (10), we can therefore compute the surface density of charge $\sigma(x, y, z)$ of the conductor. From the surface density, we can also find the charges located on the conductor itself by

$$q = \int \sigma\, dS\,. \tag{11}$$

2 Mean value theorem

Let's look to an important property of the solutions of Laplace's equation (which are called *harmonic functions*). The proof does not use Coulomb's law as in the book of Griffiths, but only Laplace's equation.

Mean value theorem: if V solves (4), then the average value that V assumes on the surface of any sphere is equal to the value that V assumes at its center.

Proof: Take a sphere of radius r centered at point C. Then the average value of V on the surface S of the sphere is

$$V_{\text{av}}(r) = \frac{1}{4\pi r^2} \int_S V(\boldsymbol{r}) dS = \frac{1}{4\pi} \int d\theta\, d\varphi\, V(r, \theta, \varphi)\, \sin\theta\,, \tag{12}$$

where we used that the surface element is $dS = r^2 \sin\theta\, d\theta d\varphi$. Notice that dS is proportional to r^2 and this cancels the same factor in the denominator. Now take the derivative with respect to r; the derivative acts only on $V(r)$

$$\frac{dV_{\text{av}}}{dr} = \frac{1}{4\pi} \int d\theta\, d\varphi\, \frac{\partial V}{\partial r}\, \sin\theta = \frac{1}{4\pi r^2} \int_S \frac{\partial V}{\partial r} dS = \frac{1}{4\pi r^2} \int_S \boldsymbol{\nabla} V \cdot d\mathbf{S} \tag{13}$$

where in the last step we have used that

$$\frac{dV}{dr} dS = \boldsymbol{\nabla} V \cdot \hat{r} dS = \boldsymbol{\nabla} V \cdot d\mathbf{S}\,. \tag{14}$$

We can now apply the divergence theorem obtaining

$$\frac{dV_{\rm av}}{dr} = \frac{1}{4\pi r^2}\int_S \boldsymbol{\nabla} V \cdot d\mathbf{S} = \frac{1}{4\pi r^2}\int_\tau \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} V)\, d\tau = \frac{1}{4\pi r^2}\int_\tau \nabla^2 V\, d\tau = 0 \tag{15}$$

where τ is the volume surrounded by S.

Since $\frac{dV_{\rm av}}{dr} = 0$ it follows that $V_{\rm av}$ is independent of the radius of the sphere where the average is performed. In particular $V_{\rm av}(r)$ is equal to the value it assumes when $r \to 0$, i.e. to the value that $V(r)$ has at the center C of the sphere. This proves the theorem.